Rationalizing Substitutions

We have seen that partial fraction decomposition provides us with a method for integrating any rational function. In this note, we will consider another situation where partial fraction decomposition can be used as an integration technique.

Suppose we begin with an integral of the form

$$\int \frac{\sqrt{x+9}}{x} dx.$$

The integrand here not a rational function (due to the square root), however we can transform it into a rational function by making a particular substitution.

Let $u = \sqrt{x+9}$, so that $u^2 = x+9$ which we can rearrange to get $x = u^2 - 9$ and $dx = 2udu$.

We may then rewrite the original integral as

$$\begin{aligned} \int \frac{\sqrt{x+9}}{x} dx &= \int \frac{u}{u^2-9}(2udu) \\ &= \int \frac{2u^2}{u^2-9} du \end{aligned}$$

We note that $\deg(P(u)) = \deg(Q(u))$, so we must apply the preliminary step in order to construct a proper rational function. By polynomial division, we obtain

$$\frac{2u^2}{u^2-9} = 2 + \frac{18}{u^2-9}$$

Factoring the denominator gives $u^2 - 9 = (u-3)(u+3)$, a product of distinct linear factors. Applying the decomposition theorem, we obtain

$$\frac{18}{(u-3)(u+3)} = \frac{3}{(u-3)} - \frac{3}{(u+3)}$$

Returning to the integral, we have

$$\begin{aligned} \int \frac{\sqrt{x+9}}{x} dx &= \int \left(2 + \frac{3}{(u-3)} - \frac{3}{(u+3)}\right) du \\ &= 2u + 3\ln|u-3| - 3\ln|u+3| + C \\ &= 2\sqrt{x+9} + 3\ln|\sqrt{x+9}-3| - 3\ln|\sqrt{x+9}+3| + C \\ &= 2\sqrt{x+9} + 3\ln\left|\frac{\sqrt{x+9}-3}{\sqrt{x+9}+3}\right| + C \end{aligned}$$