# 1 The Mean Value Theorem

**Definition.** *Suppose $f(x)$ is a function defined on a domain $D$. We say that $f$ has a* **global (or absolute) maximum** *at a point $c \in D$ if $f(c) \geq f(x)$ for all $x \in D$.*

*We say that $f$ has a* **global (or absolute) minimum** *at $d \in D$ if $f(d) \leq f(x)$ for all $x \in D$.*

*The numbers $f(c)$ and $f(d)$ are called the* **global maximum/minimum values** *of $f$ on the domain $D$.*

For a constant function $f(x) = b$, every point $x_0 \in \mathbb{R}$ satisfies both of these inequalities, and so every point $x_0 \in \mathbb{R}$ is both an absolute maximum and absolute minimum of $f$.

On the other hand, if we consider the function $g(x) = x^3$ on its domain $D = \mathbb{R}$, then $g$ has neither an absolute maximum nor an absolute minimum. This will also be true if we consider $g(x)$ on any open interval $(a, b) \in \mathbb{R}$. For closed intervals however, we have the following result.

**Theorem 1.1.** *If $f(x)$ is continuous on a closed interval $[a, b]$, then $f(x)$ attains a maximum and minimum value.*

We state this theorem without proof for now, but will return to it later in the course.

**Definition.** *The function $f$ defined on a domain $D$ has a* **local (or relative) maximum** *at a point $c \in D$, if there is an open interval $I \subset D$ such that $c \in I$ and $f(c) \geq f(x)$ for all $x \in I$.*

*The function $f$ has a* **local (or relative) minimum** *at $d \in D$ if there is an open interval $J \in D$ such that $c \in J$ and $f(d) \leq f(x)$ for all $x \in J$.*

Collectively, we refer to maxima and minima as **extrema**, or extreme points.

**Lemma 1.2.** *Let $f(x)$ be a differentiable function on an interval $(a, b)$. Suppose $x_0 \in (a, b)$. If $f'(x_0) > 0$, then for $x < x_0$ close to $x_0$ we have $f(x) < f(x_0)$ and for $x > x_0$ and close to $x_0$ we have $f(x) > f(x_0)$.*

In other words, if $f'(x_0) > 0$, then $f(x)$ is an increasing function *near* $x_0$. We can make a similar statement if $f'(x_0) < 0$ (i.e. $f$ is decreasing near $x_0$).

*Proof.* By definition, $f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$. If $f'(x) > 0$, then there exists an interval $(x_0 - \delta, x_0 + \delta)$ such that

$$\frac{f(x) - f(x_0)}{x - x_0} > 0, \text{ for } x \neq x_0$$

Suppose first that $x_0 < x < x_0+\delta$. Then $x-x_0 > 0$, and so we must have that $f(x)-f(x_0) > 0$, hence $f(x) > f(x_0)$. On the other hand, if $x_0 - \delta < x < x_0$, then the same inequality shows that $f(x) < f(x_0)$. □

**Theorem 1.3** (Fermat's Theorem)**.** *Let $f(x)$ be defined on an open interval $[a, b]$, and suppose that $f(x)$ attains a maximal (or minimal) value at a point $c \in (a, b)$. If $f(x)$ is differentiable at $x = c$, then $f'(c) = 0$.*

*Proof.* We can assume that $c$ is a maximum of $f(x)$, since the case when $c$ is a minimum can be done in a similar way. We will arguer by contradiction and suppose that $f'(x) \neq 0$.

Then either $f'(c) > 0$ or $f'(x) < 0$. If $f'(c) > 0$, then by the above Lemma, we know that $f(x) > f(c)$ for $x > c$ close enough to $c$ (this contradicts $c$ being a maximum).

If on the other hand $f'(c) < 0$, then we know that $f(x) > f(c)$ for $x < c$ close enough to $c$. This also contradicts $c$ being a maximum for $f$.

This contradiction proves the theorem, since we have shown that the only way $c$ can be a maximum for $f$ is if $f'(c) = 0$. □

**Definition.** *A point $c$ is called a* **critical point** *of a differentiable function $f$ if $f'(c) = 0$.*

With this terminology we can restate Fermat's theorem: If $c$ is a local maximum or minimum of $f$, then $c$ is a critical point of $f$.

We should be careful with this statement, as the converse does not hold. Just because $c$ is a critical point for $f$, $c$ is not necessarily a local max or min for $f$.

**Theorem 1.4** (Rolle's Theorem)**.** *Suppose $f(x)$ is continuous on the interval $[a, b]$, differentiable on $(a, b)$ and $f(a) = f(b)$. Then there exists a number $c \in (a, b)$ such that $f'(c) = 0$.*

*Proof.* We know that since $f$ is continuous on $[a, b]$, it attains its maximum value, say $M$, and its minimum value, say $m$. If both of these extrema are attained at endpoints of the interval, then we must have $M = m$, since $f(a) = f(b)$.

So in this case, $f$ must be constant on the interval $[a, b]$ (every $y$-value is bounded below and above by the same number), and so $f'(x) = 0$ for all $x \in (a, b)$.

If we are not in this case, then we must have $M > m$, and at least one of the exteme values is attained at a point $c \in (a, b)$ (an interior point). By Fermat's theorem, we must have $f'(c) = 0$. □

We now have all the necessary ingredients to state and prove the Mean Value Theorem.

**Theorem 1.5** (Mean Value Theorem)**.** *Suppose that $f(x)$ is continuous on $[a, b]$ and differentiable on $(a, b)$. Then there exists a point $c \in (a, b)$ such that*

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

*Proof.* We define another function

$$F(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

This function satisfies the conditions for Rolle's theorem: it is continuous on $[a, b]$ because it isa difference of a continuous function $f(x)$ and a linear (hence continuous) function

$$f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

On the interval $(a, b)$ we have

$$F'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}.$$

Finally, we have

$$\begin{aligned} F(a) &= f(a) - f(a) \\ &= 0 \end{aligned}$$

$$\begin{aligned} F(b) &= f(b) - f(a) - \frac{f(b) - f(a)}{b - a}(b - a) \\ &= f(b) - f(a) - (f(b) - f(a)) \\ &= 0 \end{aligned}$$

So, $F(a) = F(b)$.

Applying Rolle's theorem to $F(x)$ tells us that there exists a point $c \in (a, b)$ such that $F'(c) = 0$.

Thus,

$$F'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} = 0 \qquad \Longrightarrow \qquad f'(c) = \frac{f(b) - f(a)}{b - a}$$

□

We can use the Mean Value Theorem to prove that content functions are the only ones have derivatives which are identically zero.

**Corollary 1.6.** *Suppose $f(x)$ is a differentiable function such that $f'(x) = 0$ for all $x$. Then $f(x)$ is a constant function.*

*Proof.* Choose any two points $a$ and $b$ in the domain of $f(x)$, with $a < b$. By the Mean Value Theorem, there exists a point $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c) = 0.$$

It follows that $f(b) = f(a)$, Since $a$ and $b$ were chosen arbitrarily, $f(x)$ is a constant function. □

## 2 Applications of the Mean Value Theorem

The Mean Value Theorem (or Rolle's Theorem) can be used to prove certain inequalities or certain properties of an equation. The questions below provide two such examples.

**Example 1.** *Prove that if $x > 0$, then $\ln(x + 1) < x$.*

*Let $a = 0$, $b = x$ and $f(x) = \ln(x + 1) - x$. Then*

$$f'(x) = \frac{1}{x + 1} - 1 = \frac{-x}{x + 1}.$$

*Applying the Mean Value Theorem to the function $f$ on the interval $[a, b] = [0, x]$, there exists a point $c \in (0, x)$ such that*

$$\begin{aligned} f'(c) &= \frac{f(x) - f(0)}{x - 0} \\ \frac{-c}{c + 1} &= \frac{(\ln(x + 1) - x)}{x} \end{aligned}$$

*Now since $c > 0$, the Left-hand side of the above equation is a negative number. This gives*

$$\frac{(\ln(x + 1) - x)}{x} < 0$$

*Under the assumption that $x > 0$, we have $\ln(x + 1) - x < 0$, and so $\ln(x + 1) < x$.*

**Example 2.** *Show that the equation $x^3 - 12x + 6 = 0$ has at most one root in the interval $[-1, 1]$.*

*Consider the function $f(x) = x^3 - 12x + 6$, and assume for contradiction that there are two distinct numbers $a, b \in [-1, 1]$ with $f(a) = f(b) = 0$. We may label them so that $a < b$.*

*The function $f$ is continuous on $[a, b]$ (since $[a, b] \subset [-1, 1]$ and $f$ is continuous on $[-1, 1]$), and $f$ is differentiable on $(a, b)$ (again since $(a, b) \subset [-1, 1]$). Furthermore, by our assumption we have $f(a) = f(b)$. Thus, $f$ satisfies the conditions for Rolle's theorem on the interval $[a, b]$.*

*Applying Rolle's theorem, there exists some* $c \in (a,b)$ *such that* $f'(c) = 0$. *Since* $f'(x) = 3x^2 - 12$, *we have*

$$\begin{aligned} 3c^2 - 12 &= 0 \\ c^2 - 4 &= 0 \\ c &= \pm 2 \end{aligned}$$

*This is a contradiction, since we assumed* $c \in (a,b)$, *and hence* $c \in (-1,1)$.