Limits of Recursively-Defined Sequences (Jan. 27)

Consider an infinite convergent sequence $\{a_n\}$ and a function $f : \mathbb{R} \to \mathbb{R}$. Recall the following theorem from class:

Theorem. *If $\lim_{n\to\infty} a_n = L$ and f is continuous at L, then $\lim_{n\to\infty} f(a_n) = f(L)$.*

This theorem provides us with a useful tool for determining the convergence of a sequence which is defined recursively (also called inductively) instead of explicitly.

Suppose $\{a_n\}$ is defined by $a_{n+1} = f(a_n)$ for a continuous function f, and assume that the sequence is convergent, i.e. $\lim a_n = L$ for a finite number L. Since

$$\lim a_n = \lim a_{n+1}$$

we can take the limit on both sides of the equation for a_{n+1}:

$$\begin{aligned} \lim_{n\to\infty} a_{n+1} &= \lim_{n\to\infty} f(a_n) \\ L &= f(L) \end{aligned}$$

This last line will thus give the value for L, provided that the equation can be solved for L.

Example 1. *Suppose the sequence $\{a_n\}$ is defined by $a_1 = 1$ and $a_{n+1} = \frac{(a_n)^2+9}{6}$. Using the method we've just described, we have*

$$\begin{aligned} L &= \frac{L^2+9}{6} \\ 6L &= L^2 + 9 \\ 0 &= L^2 - 6L + 9 \\ L &= 3 \end{aligned}$$

Warning: This process will only yield the correct answer if we already know that $\{a_n\}$ is a convergent sequence. While we might be able to solve for L in the resultant equation, if the sequence is divergent this solution will be incorrect.

Example 2. *Consider the Fibonacci sequence $\{f_n\}$, which is defined by $f_1 = 1, f_2 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for all $n \geq 3$. If we attempted to apply this process assuming that $\lim_{n\to\infty} f_n = L$ for some number L, we would obtain*

$$\begin{aligned} \lim_{n\to\infty} f_n &= \lim_{n\to\infty} (f_{n-1} + f_{n-2}) \\ L &= L + L \\ L &= 0 \end{aligned}$$

But of course we know that this is not possible, since the sequence diverges to infinity.

For another example of how this process can go horribly wrong for divergent sequences, see exercise 4.9 in Prof. Shafikov's notes on Sequences.