An Introduction to Mathematical Induction

One of the most important goals of mathematics is to discover and characterize patterns and sequences. The main mathematical tool used to prove statements about patterns or sequences is **mathematical induction** (also referred to as just induction). Induction is also an essential tool in computer science and programming.

We will start with a short analogy for the process of induction:

Suppose you are climbing an infinitely high staicase, and you want to know whether you will be able to reach an arbitrarily high step. Well, suppose you can make the following two statements about your climbing abilities:

1. I can climb the first step.
2. Once I am standing on any step, I can always climb to the next step.

If both of these statements are true, then by the first one you know that you can climb the first step, and by the second you know you can keep climbing to the second, then the third, and there is no reason you should ever have to stop. Hence you know that you can climb up to any step on the staircase.

Notice however that we need BOTH of these statements in order to ge the desired result. If I only have the first statement, then all I know is that I can climb the first step. After that I may be stuck. On the other hand, if I only have the second statement, then I don't know whether I can actually make it to the first step of the staircase.

Let $P(n)$ be a logical statement (usually involving $=$, $<$, $\leq$ or $\neq$). We can break down the process of induction into proving two statements:

1. $P(1)$ is true
2. For any integer $k > 0$, if $P(k)$ is true, then $P(k+1)$ is true.

By proving BOTH of these statements, we have thus proved that $P(n)$ is true to all integers $n > 0$.

In our staircase example, the statement $P(n)$ is "I can climb to the n-th step".

So, $P(1)$ can be read as: "I can climb the first step", and the implication $P(k) \implies P(k+1)$ can be read as: "Once I have climbed to the kth step, I can climb to the $k+1$st step.

A proof by induction involves two steps:

1. **Base Case**: Prove that $P(1)$ is true.
2. **Inductive Step**: Assume that $P(k)$ is true for an arbitrary integer $k > 0$ (this assumption is called the **inductive hypothesis**) and prove that $P(k+1)$ is true.

Example 1. *Show that* $1+2+\cdots+n=\frac{n(n+1)}{2}$

Proof. Let $P(n)$ be the statement: $1+2+\cdots+n=\frac{n(n+1)}{2}$.

BASE CASE: Show that $P(1)$is true:

We have $LHS=1$. Looking at the Right-hand side we have:

$$RHS=\frac{1(1+1)}{2}=\frac{2}{2}=1$$

Thus, $LHS=RHS$ so $P(1)$ is true.

INDUCTIVE STEP: Suppose $P(k)$ is true for some $k>0$. That is,

$$1+2+\ldots k=\frac{k(k+1)}{2} \qquad \text{(Inductive Hypothesis)}$$

Show that $P(k+1)$ is true:

We have

$$\begin{aligned} LHS &= 1+2+\cdots+(k+1) \\ &= (1+2+\cdots+k)+(k+1) \\ &= \frac{k(k+1)}{2}+(k+1) \quad \text{(by IH)} \\ &= \frac{k(k+1)+2(k+1)}{2} \\ &= \frac{k^2+3k+2}{2} \end{aligned}$$

Now, looking at the other side of the equation, we have

$$\begin{aligned} RHS &= \frac{(k+1)((k+1)+1)}{2} \\ &= \frac{(k+1)(k+2)}{2} \\ &= \frac{k^2+3k+2}{2} \end{aligned}$$

Thus, $LHS=RHS$, so $P(k+1)$ is true.

Therefore, by the process of mathematical induction, we have shown that

$$1+2+\cdots+n=\frac{n(n+1)}{2} \text{ for all } n>0.$$

□

Example 2. *Show that* $n! > 2^n$ *for all* $n \geq 4$

Proof. Let $P(n)$ be the statement: $n! > 2^n$.

BASE CASE: Show that $P(4)$ is true:

We have
$$LHS = 4! = 24 \text{ and } RHS = 2^4 = 16.$$
Since $24 > 16$ we have $LHS > RHS$, so $P(4)$ is true.

INDUCTIVE STEP: Suppose $P(k)$ is true for some $k \geq 4$. That is,
$$k! > 2^k \qquad \text{(Inductive Hypothesis)}$$
Show that $P(k+1)$ is true:

We have
$$\begin{aligned} LHS &= (k+1)! \\ &= 1 \cdot 2 \cdots k \cdot (k+1) \\ &= k!(k+1) \\ &> 2^k(k+1) \quad \text{(by IH)} \end{aligned}$$

Now since $k \geq 4$, we know that $k+1 \geq 4$ and so $k+1 > 2$. So we have
$$\begin{aligned} LHS &> 2^k(k+1) \\ &> 2^k \cdot 2 \\ &= 2^{k+1} \\ &= RHS \end{aligned}$$

Thus, $LHS > RHS$, so $P(k+1)$ is true.

Therefore, by the process of mathematical induction, we have shown that
$$n! > 2^n \text{ for all } n \geq 4$$

□